Kinetic approach for the ion drag force in a collisional plasma

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The linear kinetic approach to calculate the ion drag force in a collisional plasma is generalized. The model collision integral (for ion-neutral collisions) is discussed and employed to calculate the plasma response for arbitrary velocity of the plasma flow and arbitrary frequency of the collisions. The derived plasma response is used to calculate the self-consistent force on the test charged particle. The obtained results are compared to those of the traditional pair collision approach, and the importance of the self-consistent kinetic consideration is highlighted. In conclusion, the applicability of the proposed approach is discussed.

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I. INTRODUCTION

The ion drag force—the momentum transfer from flowing ions to charged microparticles embedded into a plasma—is an inevitable and exceptionally important factor in dusty (complex) plasmas. Ion flows are usually induced due to "global" large-scale electric fields always existing in plasmas (e.g., ambipolar or sheath fields). Knowledge of the ion drag force as a function of the plasma parameters (which may vary over a quite broad range) is necessary in many complex plasma experiments [1–6]. The traditional way to derive the ion drag force on the test charged particle—the so-called "pair collision approach"—is based on the solution of the mechanical problem of the ion motion in the field of the particle. Having the ion trajectories calculated, the force is then obtained as the momentum transfer averaged over a given velocity distribution of ions. Initially, the pair collision approach was applied in Refs. [7,8] to calculate ion drag in the "Coulomb scattering" limit—basically, this is the linear approximation assuming ion scattering with small angles within the Debye sphere. Recently, the approach was developed by Khrapak *et al.* [9,10] to take into account large-angle scattering.

The pair collision approach is intrinsically inconsistent. There are the following reasons for that: (i) While the ion interacts with the charged particle, the interactions with other species (in particular, ion-neutral collisions) are *neglected*. Actually, this basic assumption gives the name to the approach. (ii) The approach *presumes* a certain potential distribution around the test charge (usually, the isotropic Debye-Hückel or Yukawa potential), although the potential is a self-consistent function of the plasma environment (e.g., ion flow velocity). (iii) The approach *presumes* certain distribution function for ions (usually, the shifted Maxwellian distribution).

All these issues can be successfully resolved by employing the *self-consistent* kinetic approach. Instead of deriving single-ion trajectories and then integrating the resulting momentum transfer, one should solve the Poisson equation coupled to the kinetic equation for ions and obtain the self-consistent electrostatic potential around the particle. The polarization electric field at the origin of the test charge gives us the force on the particle. As long as the linear approximation is applicable (the so-called "linear dielectric response formalism," e.g., [11]), the whole problem is basically reduced to the calculation of the appropriate plasma response function (permittivity). Recently, Ivlev *et al.* [12] proposed to use this formalism to calculate the ion drag force in the plasma with subthermal flow, which allowed us to take into account ion-neutral collisions.

In this paper we generalize the linear kinetic approach proposed in Ref. [12]: We discuss applicability of the model collision integral for ions (Sec. II) and calculate the self-consistent plasma response for arbitrary velocity of the plasma flow and arbitrary frequency of the ion-neutral collisions (Sec. III). The latter allows us to calculate self-consistently the ion drag force on the test charged particle (Sec. IV). In conclusion (Sec. V), we discuss the applicability of the proposed approach, compare the obtained results with the results from the pair collision approach, and demonstrate the importance of the self-consistent kinetic consideration.

II. KINETIC EQUATION

In many cases, collisions in weakly ionized plasmas can be included into consideration in the form of a *model* collision integral, which is expressed in terms of linear (often algebraic) operators [13–16]. It cannot be derived rigorously from the Boltzmann integral: The particular functional form of the model integral is based on phenomenology and fulfills certain conservation laws (e.g., particle number, momentum, energy, etc.). Although the model integrals cannot describe processes accompanying collisions "precisely," such an approach (being properly chosen for a particular problem) usually allows us to avoid unnecessary mathematical complexity related to the treatment of the Boltzmann integral and, at the same time, retains the physical essence of the considered process.

For the ion-neutral collisions we propose to employ the model collision integral in the Bhatnagar-Gross-Krook (BGK) form [17,18]. Then the kinetic equation for ions is

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{e\mathbf{E}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}} = \nu(n\Phi - f), \tag{1}$$

where $\Phi(v)=(2\pi v_T^2)^{-3/2}\exp(-v^2/2v_T^2)$ is the (isotropic) Maxwellian velocity distribution of neutrals normalized to

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unity, $v_T=\sqrt{T/m}$ is the thermal velocity of neutrals, and $n=\int f d\mathbf{v}$ is the ion density. The collision operator is proportional to the effective frequency of the ion-neutral collisions, ν, which is assumed to be *constant*. The charge-exchange collisions are usually dominant in typical low-density discharge plasmas (for ions in their parent gases). For this type of collisions, each "eliminated" ion is substituted by an ion "created" from a neutral—exactly what the right-hand side of Eq. (1) stands for. In a homogeneous plasma without external perturbation we obviously have $f=n\Phi$. Hence, the *functional form* of the BGK approach is particularly suitable for the description of the charge-exchange collisions. We note that in reality the collision cross section is a rather complicated monotonically decreasing function of the ion velocity which cannot be generally approximated by any simple scaling [14,19]. It is reasonable, therefore, to choose the approximation $\nu=$const which allows us to employ the model collision operator in the convenient form of Eq. (1).

"Unperturbed" distribution

In order to calculate the force on the test charged particle embedded into a flowing plasma, we have to obtain the self-consistent electric field $\mathbf{E}_p$ induced in the plasma by the particle. The plasma flow is caused by a *global* electric field $\mathbf{E}_0$ which is assumed to be known. Then the total electric field is $\mathbf{E}=\mathbf{E}_0+\mathbf{E}_p$. The distribution function and the density are $f=f_0+f_p$ and $n=n_0+n_p$, respectively, with f_0 determined by the "unperturbed" equation

$$u\frac{\partial f_0}{\partial v_\parallel}=n_0\Phi-f_0, \tag{2}$$

where the subscript $\parallel$ denotes the direction along the global field (which is supposed to be homogeneous), n_0 is the ambient (constant) ion density, and $u=eE_0/m\nu$ is the ion flow velocity in the mobility limit. Equation (2) readily yields the "unperturbed" distribution

$$f_0(v_\parallel,v_\perp)=\frac{n_0\Phi_\perp}{u}\int_{-\infty}^{v_\parallel}\Phi_\parallel(v'_\parallel)\exp\left(-\frac{v_\parallel-v'_\parallel}{u}\right)dv'_\parallel, \tag{3}$$

where we introduced longitudinal and transverse factors of the neutral velocity distribution, $\Phi_\parallel(v_\parallel)=(2\pi v_T^2)^{-1/2}\exp(-v_\parallel^2/2v_T^2)$ and $\Phi_\perp(v_\perp)=(2\pi v_T^2)^{-1}\exp(-v_\perp^2/2v_T^2)$, respectively, so that $\Phi\equiv\Phi_\parallel\Phi_\perp$. Without the global field—i.e., in the limit $u\to 0$—we have $f_0\to n_0\Phi$. As long as the plasma flow is subthermal the deviation of Eq. (3) from the shifted Maxwellian distribution remains small, $f_0\simeq n_0\Phi(v)(1+uv_\parallel/v_T^2)$. However, for $u\gtrsim v_T$ the deviation is significant. One can easily see this, e.g., in the limit of cold ions, $T\to 0$, when Eq. (3) tends to $f_0\propto\delta(\mathbf{v}_\perp)\exp(-v_\parallel/u)$ for $v_\parallel\geqslant 0$ (and $f_0=0$ for $v_\parallel<0$), instead of $f_0\propto\delta(\mathbf{v}-\mathbf{u})$.

III. PLASMA RESPONSE

A self-consistent electric field of the test particle, $\mathbf{E}_p$, can be obtained from the solution of Eq. (1) coupled to the Poisson equation. In the linear approach, one should derive the response function of the plasma—permittivity ε—which determines the Green function of the Poisson equation in the plasma and, hence, yields $\mathbf{E}_p$. Let us first derive the ion response. As usual, we present the perturbations in the form $f_p\propto\exp(-i\omega t+i\mathbf{k}\cdot\mathbf{r})$ for the distribution function and $\mathbf{E}_p=-\nabla\varphi_p=-i\mathbf{k}\varphi_p$ for the field. We assume f_p to be much smaller than f_0 (this automatically provides $n_p\ll n_0$), whereas the ratio $|\mathbf{E}_p/\mathbf{E}_0|$ can be arbitrary. Then we obtain, from Eq. (1),

$$-i(\omega-\mathbf{k}\cdot\mathbf{v})f_p+\nu\left(u\frac{\partial f_p}{\partial v_\parallel}+f_p\right)=\nu n_p\Phi+i\frac{e\varphi_p}{m}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{v}}, \tag{4}$$

with f_0 from Eq. (3). The principal difference of Eq. (4) from the conventionally used equation for the perturbed ion distribution (which is employed to derive the ion response [18]) is that the latter usually does not take into account deviations of f_0 from the equilibrium (in our case, Maxwellian) distribution. As we already mentioned in the previous section, the unperturbed distribution f_0 can be far from equilibrium due to the presence of the global electric field $\mathbf{E}_0$. This fact is taken into account on the right-hand side of Eq. (4). Also, an extra term $\nu u(\partial f_p/\partial v_\parallel)$ appears in the equation, due to the presence of the global field. This term contributes to the collisional broadening of the Landau resonance, in addition to the conventional term νf_p.

The plasma permittivity is $\varepsilon=1+\chi_e+\chi_i$. For the electron contribution we use the Boltzmann response $\chi_e\simeq(k\lambda_{D_e})^{-2}$. The ion susceptibility follows from the Poisson equation $\chi_i=-(4\pi e/k^2)(n_p/\varphi_p)$. We obtain the relation between $n_p=\int f_p d\mathbf{v}$ and φ_p by integrating the solution of Eq. (4) over the velocity space. Introducing the new variable $d\eta=u^{-1}dv_\parallel$, after some routine algebra we derive the ion susceptibility in the following form: $\chi_i=(v_T/\lambda_{Di}\nu)^2\mathcal{J}_2/(1-\mathcal{J}_1)$. The ion Debye length λ_{Di} corresponds to the unperturbed density n_0, and the functions $\mathcal{J}_{1,2}$ are:

$$\mathcal{J}_1(\omega,\mathbf{k})=\int_0^\infty\exp[-\Psi_{\omega,\mathbf{k}}(\eta)]d\eta,$$

$$\mathcal{J}_2(\omega,\mathbf{k})=\int_0^\infty\frac{\eta\exp[-\Psi_{\omega,\mathbf{k}}(\eta)]}{1+i(k_\parallel u/\nu)\eta}d\eta. \tag{5}$$

The function $\Psi_{\omega,\mathbf{k}}$ in Eq. (5) is given by $\Psi_{\omega,\mathbf{k}}=(1-i\omega/\nu)\eta+\frac{1}{2}[ik_\parallel u/\nu+(kv_T/\nu)^2]\eta^2$.

Representation of the ion response via the Maxwellian dispersion function

In principle, the problem of the plasma permittivity is formally solved: The integrals in Eq. (5) rapidly converge and can easily be calculated numerically or evaluated analytically in asymptotic cases. However, one can transform the obtained results into a different, mathematically identical, but physically much more sensible form. Using the well-known representation for the dispersion function of the *Maxwellian plasma*, $\mathcal{F}(\xi)=2i\xi e^{-\xi^2}\int_{-\infty}^{i\xi}e^{-\eta^2}d\eta$ [18,20], we express the first integral in Eq. (5) in terms of $\mathcal{F}$,

$$\mathcal{J}_1(\omega,\mathbf{k}) = -\frac{i\nu}{\omega+i\nu}\mathcal{F}(\xi_1),$$

$$\xi_1 = \frac{(\omega+i\nu)/\sqrt{2}kv_T}{\sqrt{1+i(k_\parallel\nu/k^2v_T)M_T}}. \tag{6}$$

Here we introduce the control parameter of deviation from the Maxwellian equilibrium—the *thermal Mach number*

$$M_T = \frac{u}{v_T},$$

which is the ratio of the flow velocity u to the ion *thermal velocity* v_T (in contrast to the usual definition of the Mach number, when u is normalized to the ion acoustic velocity). The variable ξ_1 is the function of M_T. In order to transform the second integral, we rewrite it in the form of a double integral, $\mathcal{J}_2=\int_0^\infty\int_0^\infty\eta\exp[-\Psi_{\omega,\mathbf{k}}(\eta)]\exp\{-[1+i(k_\parallel u/\nu)\eta]x\}d\eta dx$, which can be expressed in terms of the averaged $\mathcal{F}$ as follows:

$$\mathcal{J}_2(\omega,\mathbf{k}) = \frac{(\nu/kv_T)^2}{1+i(k_\parallel\nu/k^2v_T)M_T}[1+\langle\mathcal{F}(\xi_2)\rangle],$$

$$\xi_2 = \frac{(\omega-k_\parallel v_TM_Tx+i\nu)/\sqrt{2}kv_T}{\sqrt{1+i(k_\parallel\nu/k^2v_T)M_T}}. \tag{7}$$

The variable ξ_2 is the function of M_T and the parameter x over which the average is performed, $\langle\cdots\rangle=\int_0^\infty\cdots e^{-x}dx$. Using representations of $\mathcal{J}_{1,2}$ from Eqs. (6) and (7), we rewrite the susceptibility in the final form

$$\chi_i(\omega,\mathbf{k}) = \frac{(k\lambda_{Di})^{-2}}{1+i(k_\parallel\nu/k^2v_T)M_T}\left[\frac{1+\langle\mathcal{F}(\xi_2)\rangle}{1+\dfrac{i\nu}{\omega+i\nu}\mathcal{F}(\xi_1)}\right]. \tag{8}$$

Equation (8) is the self-consistent ion susceptibility in a collisional plasma with electric field. For $E_0=0$ (i.e., $M_T=0$), it reduces to the well-known expression for the Maxwellian plasma (see, e.g., [18]): The variables $\xi_{1,2}$ tend to $\xi=(\omega+i\nu)/\sqrt{2}kv_T$ and, correspondingly, $\langle\mathcal{F}(\xi_2)\rangle\to\mathcal{F}(\xi)$.

Let us discuss how the plasma flow (electric field E_0) affects the ion response χ_i. Equation (3) shows that the unperturbed distribution function f_0 is anisotropic for finite M_T. This is not just a shift in the velocity space, but also the collisional anisotropy: The ion mean free path in the longitudinal direction increases with the Mach number and for $M_T\gtrsim 1$ scales as $\sim u/\nu\equiv M_T\ell$, whereas in the transverse direction it remains constant and equal to ℓ (where $\ell=v_T/\nu$ is the "thermal" or "isotropic" mean free path at $M_T=0$). Therefore, the terms proportional to $k_\parallel M_T$ in Eq. (8) represent the contribution of the collisional anisotropy. (Of course, this particular form of the anisotropy is peculiar to the case $\nu=\text{const}$ considered here, and it should be different if ν is velocity dependent.) Another feature of Eq. (8) is that the ion response is determined by the *averaged* Maxwellian dispersion function. The reason is the following: The function f_0 can be considered as a "superposition" of shifted Maxwellian distribution functions with "weights" $u^{-1}\exp(-v_\parallel/u)dv_\parallel\equiv e^{-x}dx$. Therefore, χ_i should be the superposition of the Maxwellian dispersion functions with the same weights—exactly what Eq. (8) stands for.

IV. ION DRAG FORCE

The self-consistent distribution of the electrostatic potential around motionless test charge eZ located at $\mathbf{r}=\mathbf{0}$ is given by the following formula [11,21]:

$$\varphi_p(\mathbf{r}) = \int\frac{4\pi eZe^{i\mathbf{k}\cdot\mathbf{r}}}{k^2\varepsilon(0,\mathbf{k})}\frac{d\mathbf{k}}{(2\pi)^3}. \tag{9}$$

A charge embedded into any anisotropic medium induces the polarization. The magnitude of the polarization field at the charge origin determines the force acting on the particle, $\mathbf{F}=-eZ\nabla\varphi_p|_{\mathbf{r}=0}$ (of course, in addition to the usual electrostatic force due to the global field, $eZ\mathbf{E}_0$). In our case the anisotropy is due to the plasma flow, and this induces the ion drag force [12,22,23]. The force is obviously parallel to the flow and can be written as

$$F = -\frac{ie^2Z^2}{\pi}\int_0^{k_{\max}}\frac{dk}{k}\int_{-k}^{k}\frac{k_\parallel dk_\parallel}{\varepsilon(0,\mathbf{k})}. \tag{10}$$

For small M_T the ion drag force was shown to diverge logarithmically at $k\to\infty$ [12]. For large M_T the divergence takes place as well. This divergence is unphysical: There exists a vicinity of the test charge, $r\lesssim R$, where the plasma perturbations induced by the charge are too strong and the linear approach is no longer valid. By an order of magnitude, R is equal to the ion Coulomb radius—the distance at which the energy of the electrostatic coupling is about the kinetic energy: i.e., $R\sim e^2|Z|/\mathcal{E}_{\text{kin}}$. For the kinetic energy we can employ the scaling $\mathcal{E}_{\text{kin}}\sim T(1+M_T^2)$, so that $R\sim R_T(1+M_T^2)^{-1}$, with $R_T=e^2|Z|/T$ the thermal (isotropic) Coulomb radius. The spatial scales $r\lesssim R$ correspond to $k\gtrsim R^{-1}$, which yields $k_{\max}\sim R_T^{-1}(1+M_T^2)$ for the upper limit of the integration. Basically, the criterium of applicability of Eq. (10) is the relative smallness of the *actual* contribution from the "nonlinear" region $r\lesssim R$. A detailed discussion of the applicability is given in Sec. V.

A. Analytic expression for $M_T\ll 1$

Substituting plasma permittivity [with ion susceptibility from Eq. (8)] into Eq. (10), one can numerically calculate the ion drag force for arbitrary collision frequency (mean free path) and Mach number (electric field), using the tabulated values of $\mathcal{F}(\xi)$ [20]. However, in the limiting cases of small and large Mach numbers analytic expressions can be obtained. Introducing the linearized plasma screening length for zero Mach number, $\lambda^{-1}=\sqrt{\lambda_{Di}^{-2}+\lambda_{De}^{-2}}$, and the dimensionless force, $\tilde{F}=F(\lambda/eZ)^2$, we expand ε into a series over small M_T and derive the ion drag force for $M_T\ll 1$ (with the so-called "logarithmic accuracy;" see Sec. V):

$$\tilde{F}\simeq\frac{1}{3}\sqrt{\frac{2}{\pi}}\left[\ln\frac{\lambda}{R_T}+\frac{1}{\sqrt{2\pi}}\mathcal{K}(\lambda/\ell)\right]M_T+O(M_T^3), \tag{11}$$

where

$$\mathcal{K}(x) = x \arctan x + \left(\sqrt{\frac{\pi}{2}} - 1\right)\frac{x^2}{1+x^2} - \sqrt{\frac{\pi}{2}} \ln(1+x^2)$$

is the "collision function." Equation (11) coincides with the formula derived recently by Ivlev *et al.* [12] using the dielectric function of the *Maxwellian* plasma. This coincidence is because f_0 [Eq. (3)] reduces to the shifted Maxwellian distribution in the limit $M_T \ll 1$ (as long as the effects linear on M_T are concerned). For $\ell \gg \lambda$ the function $\mathcal{K}$ is negligibly small compared to the Coulomb logarithm and Eq. (11) yields the standard collisionless expression for the ion drag force derived from the pair collision approach in the linear approximation (see, e.g., [7–9]). In terms of the ion kinetics, the origin of this force is the Landau damping. In the opposite limit $\ell \ll \lambda$ the hydrodynamic effects become more important, so that the mean free path rather than the screening length starts playing a role of the spatial scale. Then λ disappears from the argument of the Coulomb logarithm, and the expression in the brackets in Eq. (11) changes from $\ln(\lambda/R_T)$ to $\ln(\ell/R_T)+\sqrt{\pi/8}(\lambda/\ell)$. If collisions become "very frequent," $\ell \lesssim R_T$, the kinetic effects disappear completely and the force can be derived from the fluid dynamics approach, resulting to $\widetilde{F} \simeq \frac{1}{6}(\lambda/\ell)M_T$. Note that in rf plasmas electrons do not contribute to the force, because the electron temperature in low-density plasmas is typically two orders of magnitude higher than the ion (neutral) temperature and, therefore, the linearized plasma screening length for $M_T \ll 1$ is determined by ions, $\lambda \simeq \lambda_{Di}$. (Note, however, that in dc plasmas the situation might be different; see Ref. [24].)

B. Analytic expression for $M_T \gg 1$

As the Mach number increases, the deviation of f_0 from the Maxwellian distribution becomes stronger. Once $M_T \gtrsim 1$, the conventional susceptibility of the Maxwellian plasma is no longer applicable for ions, and Eq. (8) should be used instead. In order to calculate the ion drag force for $M_T \gg 1$, mathematically it is more convenient to employ Eq. (5). The integrals $\mathcal{J}_{1,2}(0,\mathbf{k})$ are determined by the function $\Psi_{0,\mathbf{k}} = \eta + \frac{1}{2}\alpha(\mathbf{k})\eta^2$, with $\alpha = k^2\ell^2 + ik_\parallel \ell M_T \equiv \alpha_1 + i\alpha_2$. The coefficient $\alpha(\mathbf{k})$ is a measure of collisionality: For $|\alpha| \ll 1$, both longitudinal ($\sim M_T\ell$) and transverse ($\sim \ell$) mean free paths are shorter than the corresponding spatial scales (in terms of $\mathbf{k}$), and the ion response is due to hydrodynamic effects. The first two terms of the power series are $\mathcal{J}_1 \simeq 1-\alpha$ and $\mathcal{J}_2 \simeq 1 - 2i\alpha_2 - 3\alpha$. In the opposite limit $|\alpha| \gg 1$, the mean free path exceeds the spatial scale at least in one direction, and then the kinetic effects (viz., Landau resonance) become crucial. The asymptotic expansion yields $\mathcal{J}_1 \simeq \sqrt{\pi/2}\alpha^{-1/2}$ and $\mathcal{J}_2 \simeq -i\sqrt{\pi/2}\alpha_2^{-1}\alpha^{-1/2}$. Therefore, we can divide the integration region in Eq. (10) into "collisional" ($|\alpha| \ll 1$) and "collisionless" ($|\alpha| \gg 1$) subregions and use the corresponding limiting expressions for $\mathcal{J}_{1,2}$. The resulting collisional contribution (hydrodynamic effects) turns out to be proportional to M_T^{-2}, whereas the collisionless part (Landau resonance) scales as $\propto M_T^{-1}$ and, hence, is more important at large Mach numbers. The integration finally yields the force for $M_T \gg 1$ (with the logarithmic accuracy):

$$\widetilde{F} \simeq \sqrt{\frac{2}{\pi}} \ln\left(\frac{4\ell M_T}{R_T}\right) M_T^{-1} + O(M_T^{-2}). \qquad (12)$$

Equation (12) does not depend explicitly on the screening length. Instead, the mean free path determines the argument of the logarithm (although the force is purely due to the kinetic effects). Physically, this is because the contribution of ions to the screening in the transverse direction decreases with M_T and eventually disappears. Therefore, the screening length increases at $M_T \gtrsim 1$ and tends asymptotically to the *electron* Debye length λ_{De} (which is much large than the ion Debye length) [25]. In turn, the mean free path in the transverse direction remains constant and equal to ℓ. Therefore, if λ_{De} exceeds ℓ (which is often the case in experiments—unless the gas pressure is too low—and was assumed in the above calculations), the latter plays a role of the transverse spatial scale at sufficiently large M_T. The reason why the collision effects are weak at large M_T is that the "total" ($\sim$ longitudinal) mean free path increases as $\sim M_T\ell$ and eventually becomes much larger than other spatial scales.

C. Numerical calculations

It is convenient to represent the normalized force $\widetilde{F}$ as a function of the Mach number and two parameters: The ratio of the (ion) screening length to the (thermal) Coulomb radius, λ/R_T, and the ratio of the screening length to the (thermal) mean free path, λ/ℓ. The first parameter is basically the argument of the Coulomb logarithm which characterizes the screening in the collisionless limit, and the second one is the measure of collisionality for ions. Figure 1 shows the ion drag force versus the Mach number for different values of these two parameters. One can see that analytic asymptotes agree fairly well with the numerical results—depending on the value of λ/ℓ, the discrepancy is $\lesssim 10\%$ at $M_T \lesssim 0.2$–0.3 [Eq. (11)] and $M_T \gtrsim 10$–20 [Eq. (12)]. At small Mach numbers, as long as the mean free path exceeds the screening length, the collisions do not affect the force, but in the strongly collisional case, $\ell \ll \lambda$, the force is increased. At large M_T the effect is opposite—the force increases with the mean free path, although this dependence is rather weak (logarithmic).

V. DISCUSSION AND CONCLUSIONS

A. Importance of the self-consistent approach

In order to highlight importance of the self-consistent approach proposed here to calculate the ion response and the drag force, let us compare the derived asymptotic expression for the force at large Mach numbers [Eq. (12)] with the scaling which follows from the Maxwellian dispersion function [18]. The latter can be formally obtained by setting first $M_T=0$ in Eq. (8) for the ion susceptibility and then calculating the force from Eq. (10) with $\varepsilon(\omega,k)$ at $\omega = -k_\parallel u \equiv -k_\parallel v_T M_T$. This readily gives us $\widetilde{F} \propto M_T^{-2} \ln M_T$ for $M_T \gg 1$, in contrast to Eq. (12). Thus, the self-consistent approach yields the force which is much larger than the results of calculations with the shifted Maxwellian distribution. Note that the traditional pair collision approach at large Mach

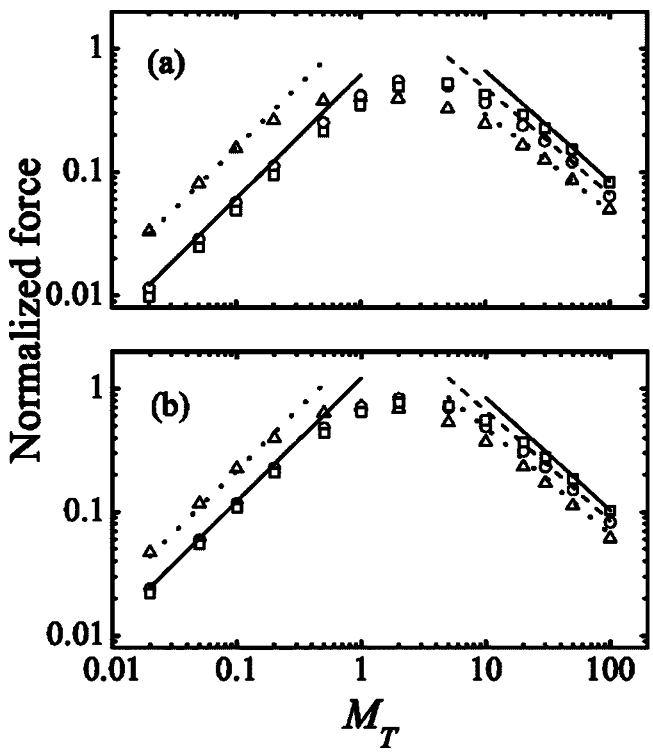

FIG. 1. Normalized ion drag force versus the thermal Mach number of the ion flow M_T. The force depends on two parameters: the ratio of the ion screening length to the thermal Coulomb radius, λ/R_T, and the ratio of the screening length to the thermal mean free path, λ/ℓ. The data points are obtained by numerical integration of Eq. (10) with the ion susceptibility from Eq. (8), for $\lambda/R_T=10$ (a) and $\lambda/R_T=100$ (b). Symbols represent $\lambda/\ell=0.1$ (□), $\lambda/\ell=1$ (○), and $\lambda/\ell=10$ (△). Analytic asymptotes at small and large Mach numbers [Eqs. (11) and (12), respectively] correspond to the same values of λ/ℓ (solid, dashed, and dotted lines, respectively).

numbers also gives the scaling $\tilde{F}\propto M_T^{-2}\ln M_T$ [8,25,26].

The reason why the power factor of the force scales as $\propto M_T^{-1}$ at large Mach numbers can be understood from Eq. (3): As we already discussed in the end of Sec. III, the unperturbed distribution f_0 is essentially a *superposition* of shifted Maxwellian functions with the weights $u^{-1}\exp(-v_\parallel/u)dv_\parallel$. Therefore, also the total force is a superposition of the forces calculated for the shifted Mawellian distributions with the corresponding weights. At $M_T\gg 1$, the contribution of ions with the longitudinal velocity $v_\parallel$ to the force scales as $\propto v_\parallel^{-2}$ [8,9] (neglecting the logarithmic factor). Hence, the superposition (viz., integration) yields the power dependence $\propto u^{-1}\int_{v_T}^{\infty}v_\parallel^{-2}\exp(-v_\parallel/u)dv_\parallel\propto M_T^{-1}+O(M_T^{-2})$ for the total force, in agreement with Eq. (12). The obtained scaling does not depend on a functional form of the weights which, in turn, are determined by a particular dependence of the collision frequency on the ion velocity (in our case, $\nu=\text{const}$). Therefore, we can presume that this scaling is a generic feature of the self-consistent approach at large Mach numbers.

Another feature of the obtained results is the dependence of the force on the ion mean free path. Figure 1 shows that frequent ion-neutral collisions ($\ell\ll\lambda$) enhance the force at small M_T. This is due to the ion focusing [12]: Each collision "eliminates" the angular momentum the ion had (with respect to the particle) before the collision. Therefore, the motion of the flowing ions becomes more "radial" due to the attraction towards the charged particle—the "focusing center" downstream moves closer to the particle. This additional focusing implies a local increase of the ion density and, hence, increase of the polarization (force). This mechanism, however, can operate only if the field of the charged particle is stronger than the global field E_0. Otherwise, if E_0 is relatively strong (Mach number is large), it should *defocus* the ion trajectories: After each collision, the ions should accelerate mostly along E_0. An increase of collisionality (decrease of ℓ) at constant $M_T\propto E_0\ell$ implies an increase of the global electric field and, hence, stronger defocusing. In turn, the latter implies the decrease of the polarization (force) which we see in Fig. 1.

It is noteworthy that the force acting on a motionless test charge embedded in a collisional flowing plasma is not equivalent to the force acting on the moving charge in a plasma at rest. The principal difference between these two cases can be easily understood in terms of the reference frames: For the moving test charge, the ions flow with respect to the charge *together* with neutrals and, hence, keep the Maxwellian distribution. In this case, the force on the charge is determined by the Maxwellian dispersion function. In contrast, when the charge is at rest and the ions flow, the neutrals remain at rest as well. Of course, this changes the ion distribution function [see Eq. (3)] and, thus, the ion drag force.

B. Remarks on the applicability

The applicability of the linear approach presumes large arguments of the logarithms in Eqs. (11) and (12)—the so-called "logarithmic accuracy" [12,21]. For $M_T\ll 1$ this is the well-known condition for the Coulomb logarithm, $\lambda/R_T\gg 1$ [9,10,21]. It is very interesting that larger Mach numbers imply *better applicability* of the linear theory—the argument of the logarithm in Eq. (12) grows with M_T. Physically this is because the range of nonlinear interaction scales as $R\propto M_T^{-2}$, so that the upper limit $k_{\max}$ in Eq. (10) increases [27,28].

The criteria of the applicability can be obtained as follows: Calculating the force we neglected the contribution of ions from the "nonlinear region" within the Coulomb radius, $r\lesssim R$. This contribution can be easily estimated in terms of the trajectories: Impact parameters of ions contributing to the polarization in this region do not exceed R. For $M_T\ll 1$ this yields logarithmically small correction, provided $\lambda/R_T\gg 1$ [12]. At $M_T\gg 1$ the "nonlinear" momentum transfer does not exceed the momentum flux $\sim mn_0v_\parallel^2R^2$. The Coulomb radius scales as $R\simeq R_T(v_T/v_\parallel)^2$ and, hence, the resulting correction to the force is less than $\sim(mn_0v_T^4R_T^2/u)\int_{v_T}^{u}v_\parallel^{-2}dv_\parallel\sim(eZ/\lambda)^2M_T^{-1}$. Thus, the contribution of nonlinear effects at large Mach numbers is logarithmically small as well [see Eq. (12)].

The calculations were performed for a pointlike particle. In reality, however, the particle has a finite size (radius) a and, therefore, a certain fraction of ions is absorbed on it: In terms of the trajectories, the ions having an impact parameter smaller than the so-called "absorption radius" ρ_{abs} transfer their momentum in direct collisions with the particle [8]. Thus, the obtained results are valid as long as the contribution of the absorbed ions to the force is small. The condition to neglect the absorbtion is that ρ_{abs} should be much smaller than R [12]. In the "collisionless" case (when the ion mean free path exceeds the spatial scale of the problem, e.g., λ), the absorption radius is determined by the "orbit-motion-limited" (OML) theory, $\rho_{abs} \simeq a\sqrt{1+R/a}$ [8,29,30]. For small Mach numbers, the (thermal) Coulomb radius is usually much larger than the particle size and, hence, the absorption does not affect the results. As the Mach number grows, the role of the ion absorption increases. This is because the Coulomb radius rapidly falls off to zero, whereas the absorption radius tends to the geometrical limit, $\rho_{abs} \simeq a$. Comparing the resulting direct momentum flux on the particle, $\sim mn_0u^2a^2$, with Eq. (12) we obtain the condition $M_T \lesssim (R_T/a)^{2/3}$ for the absorption to be neglected. Typically, $R_T/a \sim 100-300$ in complex plasmas [9], so that the absorption does not play noticeable role up to $M_T \lesssim 30-50$. Note also that when R becomes smaller than a, but the absorption still does not affect the force—i.e., at $(R_T/a)^{1/2} \lesssim M_T \lesssim (R_T/a)^{2/3}$—the particle size rather than the Coulomb radius determines the upper limit of integration in Eq. (10): i.e., $k_{max} \sim a^{-1}$. In the "collisional" case (when the ion mean free path becomes comparable with λ) the absorption changes: ρ_{abs} increases with ν [31–33]. Nevertheless, for large Mach numbers this does not affect the linear theory, since the ion motion is basically collisionless (see Sec. IV). For $M_T \ll 1$, the ion absorption does not change the results of the linear theory as well [12]. (Of course, the absorption itself changes the grain charge and, hence, implicitly affects the force.) The ion-neutral collisions also cause the ion trapping in the vicinity of the grain [32,33], which can strongly change the potential distribution around the particle and thus influence the ion drag. This, however, is an essentially nonlinear process and cannot be considered in the framework of the linear theory.

C. Conclusions and unresolved issues

The linear kinetic approach and the pair collision approach are complementary to some extent: The pair collision approach is more suitable to describe *highly nonlinear collisionless* cases when both the ion Coulomb radius and the mean free path exceed the spatial scale of the problem (e.g., λ) [9,10]. This situation is typical for subthermal ion flows, when $R \simeq R_T$ is large compared to the screening length. Small Mach numbers also imply weak distortion of the potential around the charged particle and weak deviation of the ion distribution from the shifted Maxwellian function. Therefore, there is no need to employ the self-consistent kinetic approach in this case. On the other hand, for suprathermal ions (when $M_T \gtrsim 1$ and the linear theory can be better applied) both the particle potential and ion distribution function are highly anisotropic, and then the self-consistent kinetic approach is necessary. And, of course, the kinetic approach should be used in strongly collisional cases.

In this paper we assume the global field E_0 to be *homogeneous*, which corresponds to the ion flow in the mobility limit [see Eqs. (2) and (3)]. This limit requires the ion mean free path ℓ to be smaller than the scale L_E of spatial variations of E_0. In principle, the kinetic equation can be solved for arbitrary ratio between L_E and ℓ. The considered limit $L_E \gg \ell$, being very useful for practical applications, is also relatively simple for calculations. It is important, however, to study the opposite "collisionless" limit $L_E \ll \ell$, when ions move over ballistic trajectories between rare collisions. This case will be considered elsewhere [25].

Another issue is the dependence of the ion collision frequency ν on the velocity. This dependence might play an important role at small Mach numbers, when the ion energy is relatively small and, hence, the mean free path rather than the collision frequency should be considered constant [14,19]. In this case, the collision operator in the kinetic equation has the integral form, which makes the algebra far more complicated and requires extensive numerical simulations.

And finally, in complex plasmas one often has to deal with the situation when the linear treatment is not possible for the ion drag force. As we mentioned above, this problem is important for small Mach numbers (e.g., bulk plasmas) when the linear approach can be applied only for sufficiently small (submicron) particles [12]. In this case, although the proposed form of the model collision integral remains valid, the self-consistent approach requires solution of the nonlinear Poisson equation coupled to the kinetic equation for ions.

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